Evolution from BCS Superconductivity to Bose Condensation: Analytic Results for the crossover in three dimensions

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We provide an analytic solution for the mean-field equations and for the relevant physical quantities at the Gaussian level, in terms of the complete elliptic integrals of the first and second kinds, for the crossover problem from BCS superconductivity to Bose-Einstein condensation of a *three*-dimensional system of free fermions interacting via an attractive contact potential at zero temperature. This analytic solution enables us to follow the evolution between the two limits in a particularly simple and transparent way, as well as to verify the absence of singularities during the evolution.

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I. INTRODUCTION

The interest in the crossover from BCS superconductivity to Bose-Einstein (BE) condensation has quite recently increased both experimentally and theoretically after the appearance of new angular resolved photoemission data on underdoped cuprate samples, which indicate the persistence of a gap in the single-particle excitation spectrum at temperatures well above the superconducting critical temperature.[1–3] Previous to these data the suggestion that superconductivity in cuprates (as well as in other "exotic" materials) might require an intermediate approach between the two (BCS and BE) limits has been emphasized especially by Uemura.[4]

From the theoretical point of view, evolution from BCS superconductivity to BE condensation has been considered originally by Nozières and Schmitt-Rink,[5] following the pioneering works by Eagles[6] and Leggett.[7] More recently, the crossover problem was addressed in Refs. 8–10, prompted by the experimental suggestion that the superconducting coherence length in cuprates is considerably shorter than in conventional superconductors. In all these works known results were recovered analytically in the two (BCS and BE) limits. However, no analytic result was obtained for the most interesting crossover region, which had thus to be treated numerically (with the sole exception of the two-dimensional case considered in Ref. 11).

In this paper we provide an analytic solution in the *three dimensional* case covering the *whole* crossover region for a number of physical quantities evaluated at the mean-field level and with the inclusion of Gaussian fluctuations, by considering a system of fermions in free space at zero temperature mutually interacting via an attractive *contact* potential. For this case we are, in fact, able to express these physical quantities in terms of the complete elliptic integrals of the first and second kinds, whose analytic properties and numerical values are extensively tabulated.[12] Our solution enables us to interpolate in a rigorous fashion between the two (BCS and BE) limits, thus avoiding the problems which occur with a full numerical approach. Although our analytical solution has been unavoidably obtained with simplifying assumptions (namely, at zero temperature and using a free-particle dispersion relation and a contact interaction potential), it might nevertheless be regarded as a reference solution with which numerical solutions obtained for more complicated cases could be compared.

The plan of the paper is as follows. In Section II we express a number of mean-field quantities (such as the chemical potential μ, the gap parameter Δ, the pair-correlation length ξ_{pair}, and the bound-state energy ϵ_o) in terms of the complete elliptic integrals of the first and second kinds with argument $\kappa^2 = (\sqrt{1 + x_o^2} + x_o)/(2\sqrt{1 + x_o^2})$ [where $x_o = \mu/\Delta$] ranging from 0 (BE limit) to 1 (BCS limit). In Section III our analysis is extended to quantities such as the phase coherence length ξ_{phase} and the sound velocity s, by considering the Gaussian fluctuations about the mean field. To keep the presentation compact, the properties of the elliptic integrals used in our treatment are summarized in Appendix A. In Appendix B we report for the sake of comparison the solution for the two-dimensional case given previously by Ref. 11.

II. ANALYTIC RESULTS AT THE MEAN-FIED LEVEL

We consider the Hamiltonian ($\hbar = 1$)

$$H = \sum_\sigma \int d\mathbf{r}\, \psi_\sigma^\dagger(\mathbf{r}) \left(-\frac{\nabla^2}{2m} - \mu \right) \psi_\sigma(\mathbf{r})$$
$$+ g \int d\mathbf{r}\, \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \psi_\uparrow(\mathbf{r}) \qquad (2.1)$$

where $\psi_\sigma(\mathbf{r})$ is the fermionic field operator with spin projection σ, m is the fermionic (effective) mass, and $g = V\Omega$ is the strength of the short-range (contact) potential between fermions with $V < 0$ (Ω being the volume occupied by the system).

At the zero temperature, the mean-field equations for the gap parameter Δ and the chemical potential μ are obtained by a suitable decoupling of the Hamiltonian (2.1) and are given by[13]

$$-\frac{1}{V} = \sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}}} \qquad (2.2)$$

$$n = \frac{N}{\Omega} = \frac{2}{\Omega} \sum_{\mathbf{k}} v_{\mathbf{k}}^2 \qquad (2.3)$$

where $\mathbf{k}$ is the wave vector, N the total number of fermions, and

$$\xi_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} - \mu, \ E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}, \ v_{\mathbf{k}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right). \qquad (2.4)$$

Owing to our choice of a contact potential, Eq. (2.2) diverges in the ultraviolet (both in two and three dimensions) and requires a suitable regularization. In three dimensions it is common practice to introduce the scattering amplitude a_s defined via the equation[8–10]

$$\frac{m}{4\pi a_s} = \frac{1}{\Omega V} + \frac{1}{\Omega} \sum_{\mathbf{k}} \frac{m}{\mathbf{k}^2} \quad , \qquad (2.5)$$

where the divergent sum on the right-hand side of Eq.(2.5) results in a finite value of a_s by letting $V \to 0$ in a suitable way. Subtracting Eq. (2.5) from Eq. (2.2) we obtain

$$-\frac{m}{4\pi a_s} = \frac{1}{\Omega} \sum_{\mathbf{k}} \left(\frac{1}{2E_{\mathbf{k}}} - \frac{m}{\mathbf{k}^2}\right) \qquad (2.6)$$

which has to be solved simultaneously with Eq. (2.3) to determine Δ and μ as functions of a_s.

In three dimensions it is convenient to introduce the following dimensionless quantities

$$\begin{cases} x^2 = \frac{k^2}{2m}\frac{1}{\Delta} \quad , \qquad x_o = \frac{\mu}{\Delta} \quad , \\ \xi_x = \frac{\xi_k}{\Delta} = x^2 - x_o \quad , \ E_x = \frac{E_k}{\Delta} = \sqrt{\xi_x^2 + 1} \end{cases} \qquad (2.7)$$

with $k = |\mathbf{k}|$, and express Eqs. (2.3) and (2.6) in the form

$$-\frac{1}{a_s} = \frac{2}{\pi}(2m\Delta)^{1/2} I_1(x_o) \qquad (2.8a)$$

$$n = \frac{1}{2\pi^2}(2m\Delta)^{3/2} I_2(x_o) \qquad (2.8b)$$

where

$$I_1(x_o) = \int_0^\infty dx\, x^2 \left(\frac{1}{E_x} - \frac{1}{x^2}\right) \qquad (2.9)$$

$$I_2(x_o) = \int_0^\infty dx\, x^2 \left(1 - \frac{\xi_x}{E_x}\right) \quad . \qquad (2.10)$$

The integrals (2.9) and (2.10) were originally considered by Eagles,[6] who evaluated them numerically as functions of the crossover parameter x_o ranging from $-\infty$ (BE limit) to $+\infty$ (BCS limit). We shall soon show that the integrals (2.9) and (2.10) can actually be evaluated in a closed form for all values of x_o. Note that $I_2(x_o) \geq 0$ while $I_1(x_o)$ can take both signs.

Before proceeding further, it is convenient to render Eqs. (2.8) dimensionless by introducing the Fermi energy $\epsilon_F = k_F^2/2m = (3\pi^2 n)^{2/3}/2m$ where k_F is the Fermi momentum. In this way Eq. (2.8b) becomes

$$\frac{\Delta}{\epsilon_F} = \left[\frac{2}{3I_2(x_o)}\right]^{2/3} \qquad (2.11)$$

and Eq. (2.8a) reduces to

$$\frac{1}{k_F a_s} = -\frac{2}{\pi}\left[\frac{2}{3I_2(x_o)}\right]^{1/3} I_1(x_o) \quad . \qquad (2.12)$$

Note that the right-hand sides of Eqs. (2.11) and (2.12) depend on x_o only. Equation (2.12) can thus be inverted to obtain x_o as a function of $k_F a_s$; from Eq. (2.11) and from $\mu/\epsilon_F = x_o \Delta/\epsilon_F$ one can then obtain the two parameters Δ/ϵ_F and μ/ϵ_F as functions of $k_F a_s$.

Alternatively, one can use $k_F \xi_{pair}$ as the independent variable in the place of $k_F a_s$, where ξ_{pair} is the characteristic length for pair correlation given by the following expression in three dimensions at the mean-field level:[10]

$$\xi_{pair}^2 = \frac{1}{m^2}\frac{\int_0^\infty dk(k^4\xi_k^2/E_k^6)}{\int_0^\infty dk(k^2/E_k^2)} = \frac{2}{m\Delta}\frac{I_3(x_o)}{I_4(x_o)} \qquad (2.13)$$

where

$$I_3(x_o) = \int_0^\infty dx \frac{x^4\xi_x^2}{E_x^6} \qquad (2.14)$$

$$I_4(x_o) = \int_0^\infty dx \frac{x^2}{E_x^2} \qquad (2.15)$$

are two additional integrals expressed in terms of the quantities (2.7). Contrary to the integrals I_1 and I_2, the integrals I_3 and I_4 are elementary and can be evaluated via the residues technique. One obtains:

$$I_3(x_o) = \frac{\pi}{16}\frac{x_1(1+x_1^4)}{(1+x_o^2)^{1/2}} \qquad (2.16)$$

$$I_4(x_o) = \frac{\pi}{2}x_1 \qquad (2.17)$$

with the notation

$$x_1^2 = \frac{\sqrt{1+x_o^2}+x_o}{2} \quad . \tag{2.18}$$

Making use of Eq. (2.11) we obtain eventually

$$(k_F\xi_{pair})^2 = \frac{\epsilon_F}{2\Delta}\frac{(1+x_1^4)}{(1+x_o^2)^{1/2}} = \frac{(1+x_1^4)}{2(1+x_o^2)^{1/2}}\left[\frac{3I_2(x_o)}{2}\right]^{2/3} . \tag{2.19}$$

In this way Eq. (2.12) can be dropped in favor of Eq. (2.19), which can be inverted to obtain x_o as a function of $k_F\xi_{pair}$.

There remains to evaluate the integrals $I_1(x_o)$ entering Eq. (2.12) and $I_2(x_o)$ entering Eqs. (2.11) and (2.19). To this end, we introduce the auxiliary integrals

$$I_5(x_o) = \int_0^\infty dx \frac{x^2}{E_x^3} \quad , \tag{2.20}$$

$$I_6(x_o) = \int_0^\infty dx \frac{x^2\xi_x}{E_x^3} \quad , \tag{2.21}$$

such that

$$I_1(x_o) = 2\left(x_o I_6(x_o) - I_5(x_o)\right) \tag{2.22}$$

$$I_2(x_o) = \frac{2}{3}\left(x_o I_5(x_o) + I_6(x_o)\right) \tag{2.23}$$

after integration by parts. The auxiliary integrals $I_5(x_o)$ and $I_6(x_o)$ can, in turn, be expressed as linear combinations of the complete elliptic integrals of the first $[F(\frac{\pi}{2},\kappa)]$ and second $[E(\frac{\pi}{2},\kappa)]$ kinds. Reduction of a generic integral of the elliptic kind to the normal Legendre's form has been treated at length in the literature.[14] Here we proceed as follows. Integration by parts gives for $I_6(x_o)$ [cf. Eq. (2.21)]:

$$I_6(x_o) = -\frac{1}{2}\int_0^\infty dx\, x\frac{d}{dx}\frac{1}{E_x} = \frac{1}{2}\int_0^\infty dx\frac{1}{E_x}$$
$$= \frac{1}{2}\int_0^\infty dx\frac{1}{(x^4 - 2x_o x^2 + x_o^2 + 1)^{1/2}}$$
$$= \frac{1}{2(1+x_o^2)^{1/4}}F(\frac{\pi}{2},\kappa) \tag{2.24}$$

where in the last line use has been made of the results of Appendix A and where $(0 \le \kappa^2 < 1)$

$$\kappa^2 = \frac{x_1^2}{(1+x_o^2)^{1/2}} \tag{2.25}$$

with x_1 given by Eq. (2.18). For $I_5(x_o)$ we obtain instead [cf. Eq. (2.20)]:

$$I_5(x_o) = \int_0^\infty dx\frac{x^2}{(x^4 - 2x_o x^2 + x_o^2 + 1)^{3/2}}$$
$$= (1+x_o^2)^{1/4}E(\frac{\pi}{2},\kappa) - \frac{1}{4x_1^2(1+x_o^2)^{1/4}}F(\frac{\pi}{2},\kappa) \tag{2.26}$$

where again use has been made of the results of Appendix A.

In conclusion, we obtain for the quantities of interest [cf. Eqs. (2.11), (2.12), and (2.19)]:

$$\frac{\Delta}{\epsilon_F} = \frac{1}{(x_o I_5(x_o) + I_6(x_o))^{2/3}} \tag{2.27}$$

$$\frac{\mu}{\epsilon_F} = \frac{\mu}{\Delta}\frac{\Delta}{\epsilon_F} = \frac{x_o}{(x_o I_5(x_o) + I_6(x_o))^{2/3}} \tag{2.28}$$

$$\frac{1}{k_F a_s} = -\frac{4}{\pi}\frac{(x_o I_6(x_o) - I_5(x_o))}{(x_o I_5(x_o) + I_6(x_o))^{1/3}} \tag{2.29}$$

$$k_F\xi_{pair} = \left(\frac{1+x_1^4}{2}\right)^{1/2}\frac{(x_o I_5(x_o) + I_6(x_o))^{1/3}}{(1+x_o^2)^{1/4}} \tag{2.30}$$

with $I_5(x_o)$ and $I_6(x_o)$ given by Eqs. (2.26) and (2.24), respectively. It is sometimes convenient to normalize the chemical potential, when negative, with respect to the bound-state energy ϵ_o of the associated two-fermion problem. In the three-dimensional case, ϵ_o can be expressed in terms of a_s whenever $a_s \ge 0$. One finds[10]

$$\epsilon_o = \frac{1}{ma_s^2} \quad , \tag{2.31}$$

so that [cf. Eq. (2.29)]

$$\frac{\epsilon_o}{\epsilon_F} = \frac{2}{(k_F a_s)^2} = \frac{32}{\pi^2}\frac{(x_o I_6(x_o) - I_5(x_o))^2}{(x_o I_5(x_o) + I_6(x_o))^{2/3}} \quad . \tag{2.32}$$
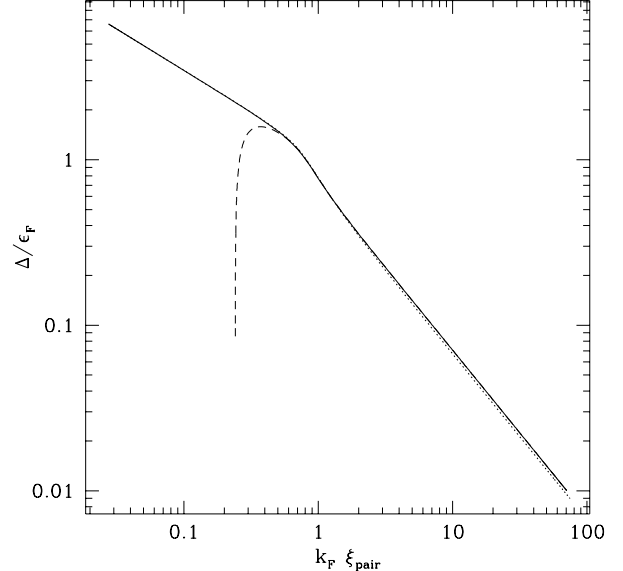

FIG. 1. Δ/ϵ_F vs $k_F\xi_{pair}$, obtained from Eqs. (2.27) and (2.30). Full curve: exact solution; dashed curve: BCS approximation, obtained by including only the first two terms in Eqs. (A.7) and (A.8); dotted curve: BE approximation, obtained by including only the first two terms in Eqs. (A.5) and (A.6).
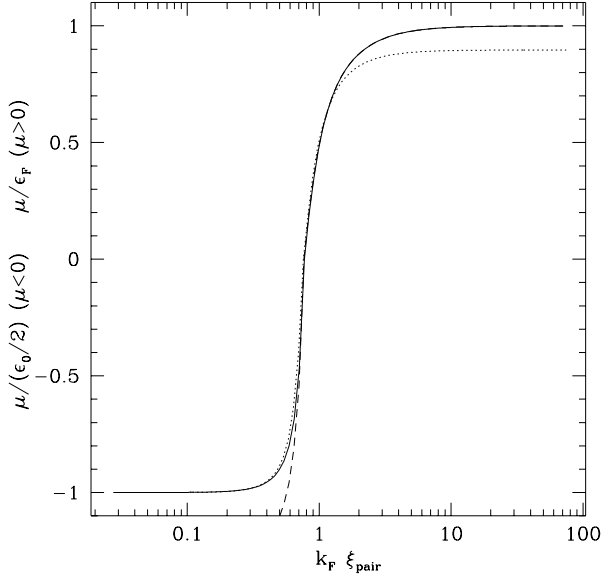
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FIG. 2. μ/ϵ_F for $\mu > 0$ and $\mu/(\epsilon_o/2)$ for $\mu < 0$ vs $k_F\xi_{pair}$, obtained from Eqs. (2.28), (2.30), and (2.32). Conventions are as in Fig. 1.
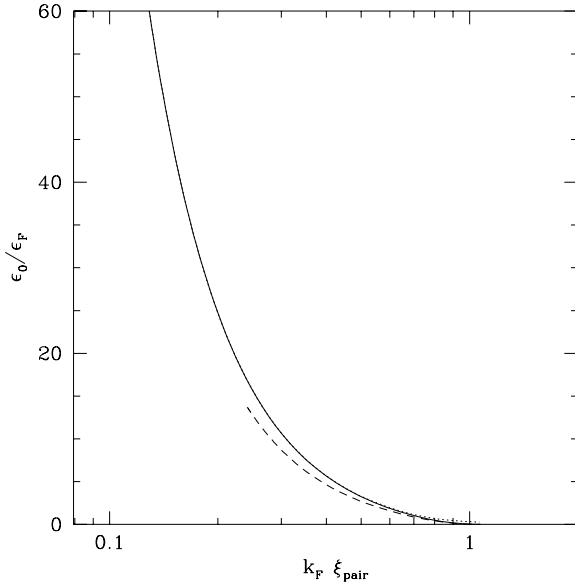

FIG. 3. ϵ_o/ϵ_F vs $k_F\xi_{pair}$, obtained from Eqs. (2.30) and (2.32) when a_s given by Eq. (2.29) is positive. Conventions are as in Fig. 1.

Numerical values of the complete elliptic integrals F and E have been extensively tabulated.[12] Otherwise, one may generate them with the required accuracy via Eqs. (A.5)-(A.8). In this way, the desired values of $I_5(x_o)$ and $I_6(x_o)$ can be obtained for given x_o (or for given $k_F\xi_{pair}$ by inverting Eq. (2.30)). In Figs. 1-3 we report, respectively, the values of Δ/ϵ_F, μ/ϵ_F for $\mu > 0$ and $\mu/(\epsilon_o/2)$ for $\mu < 0$, and ϵ_o/ϵ_F vs $k_F\xi_{pair}$ obtained by this procedure. Within numerical accuracy, all values

coincide with those calculated by solving numerically the gap equation and the normalization condition (2.3) for the limiting case of a contact potential.[10]

Besides the exact result (full curve), Figs. 1-3 show for comparison two additional curves obtained by approximating, respectively, the elliptic integrals F and E by the first two terms of Eqs. (A.5) and (A.6) (dotted curve) and by the first two terms of Eqs. (A.7) and (A.8) (dashed curve). In principle, these approximate results are expected to be reliable in the BE and BCS limits, in the order. Note, however, that the BE approximate result is surprisingly accurate on the BCS side of the crossover.

In the BCS and BE limits the values of $I_5(x_o)$ and $I_6(x_o)$ can be obtained by retaining only a few significant terms in the expansions (A.5)-(A.8). In particular, in the BCS limit $x_o \gg 1$ so that $\kappa^2 \simeq 1 - 1/(4x_o^2)$ and

$$\begin{cases} I_5(x_o) \simeq \sqrt{x_o} \\ I_6(x_o) \simeq \ln x_o/(2\sqrt{x_o}) \end{cases} . \qquad (2.33)$$

Then:

$$\begin{cases} \Delta/\epsilon_F \simeq 1/x_o \\ \mu/\epsilon_F \simeq 1 \\ 1/(k_F a_s) \simeq -(2/\pi)\ln x_o \\ k_F\xi_{pair} \simeq x_o/\sqrt{2} \end{cases} \qquad (2.34)$$

In the BE limit, on the other hand, $x_o < 0$ and $|x_o| \gg 1$ so that $\kappa^2 = 1/(4x_o^2)$ and

$$\begin{cases} I_5(x_o) \simeq \pi/(16|x_o|^{3/2}) \\ I_6(x_o) \simeq \pi/(4|x_o|^{1/2}) \end{cases} . \qquad (2.35)$$

Then:

$$\begin{cases} \Delta/\epsilon_F \simeq [16/(3\pi)]^{2/3}|x_o|^{1/3} \\ \mu/\epsilon_F \simeq -[16/(3\pi)]^{2/3}|x_o|^{4/3} \\ 1/(k_F a_s) \simeq [16/(3\pi)]^{1/3}|x_o|^{2/3} \\ \epsilon_o/\epsilon_F \simeq 2[16/(3\pi)]^{2/3}|x_o|^{4/3} \\ k_F\xi_{pair} \simeq \frac{1}{\sqrt{2}}[16/(3\pi)]^{-1/3}|x_o|^{-2/3} \end{cases} \qquad (2.36)$$

The limiting BCS (2.34) and BE (2.36) values coincide with those calculated previously by different methods.[10]

We mention, finally, that another quantity which can be evaluated analytically at the mean-field level for all values of x_o is the single-particle density of states.

All the above results hold for the three-dimensional system. Analogous results for the two-dimensional system are straightforwardly expressed in terms of elementary integrals and are reported for comparison in Appendix B.

III. ANALYTIC RESULTS AT THE GAUSSIAN LEVEL

Besides the quantities of Section II defined at the mean-field level, additional quantities whose definition

requires the introduction of Gaussian fluctuations can also be expressed analytically in three dimensions at zero temperature for the Hamiltonian (2.1), in terms of the complete elliptic integrals F and E for *all* values of the parameter x_o (*i.e.*, following the evolution from BCS to BE). In particular, we shall consider the phase coherence length ξ_{phase} (associated with the spatial fluctuations of the superconducting order parameter) and the sound velocity s (associated with the Goldstone mode of the broken symmetry).

The matrix of the Gaussian fluctuations has elements[10,15]

$$\boldsymbol{\Gamma}(\mathbf{q},\omega) = \begin{pmatrix} A(\mathbf{q},\omega) & B(\mathbf{q},\omega) \\ B(\mathbf{q},\omega) & A(-\mathbf{q},-\omega) \end{pmatrix} \tag{3.1}$$

where ω is the frequency and (for a real order parameter)

$$A(\mathbf{q},\omega) = \frac{1}{\Omega}\sum_{\mathbf{k}}\left(\frac{1}{2E_{\mathbf{k}}} - \frac{u_{\mathbf{k}}^2 u_{\mathbf{k}-\mathbf{q}}^2}{E_{\mathbf{k}} + E_{\mathbf{k}-\mathbf{q}} - \omega - i\eta} - \frac{v_{\mathbf{k}}^2 v_{\mathbf{k}-\mathbf{q}}^2}{E_{\mathbf{k}} + E_{\mathbf{k}-\mathbf{q}} + \omega + i\eta}\right) \tag{3.2}$$

$$B(\mathbf{q},\omega) = \frac{1}{\Omega}\sum_{\mathbf{k}} u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{k}-\mathbf{q}}v_{\mathbf{k}-\mathbf{q}}\left(\frac{1}{E_{\mathbf{k}} + E_{\mathbf{k}-\mathbf{q}} - \omega - i\eta} + \frac{1}{E_{\mathbf{k}} + E_{\mathbf{k}-\mathbf{q}} + \omega + i\eta}\right) \tag{3.3}$$

at zero temperature. In these expressions, $u_{\mathbf{k}}^2 = 1 - v_{\mathbf{k}}^2$ and η is a positive infinitesimal. To extract ξ_{phase} and s we need consider the expansion of $A(\mathbf{q},\omega)$ and $B(\mathbf{q},\omega)$ for small values of $|\mathbf{q}|$ and ω, such that

$$\frac{\mathbf{q}^2}{2m} \ll \omega^* \qquad \text{and} \qquad \omega \ll \omega^* \tag{3.4}$$

where

$$\omega^* = \begin{cases} 2\Delta, & \mu > 0 \\ 2\sqrt{\Delta^2 + \mu^2}, & \mu < 0 \end{cases} \tag{3.5}$$

The result is:

$$A(\mathbf{q},\omega) = a_0 + a_1\omega + a_2\mathbf{q}^2 + a_3\omega^2 + \ldots \tag{3.6}$$
$$B(\mathbf{q},\omega) = b_0 + b_2\mathbf{q}^2 + b_3\omega^2 + \ldots, \tag{3.7}$$

with

$$a_0 = \frac{1}{\Omega}\sum_{\mathbf{k}}\frac{\Delta^2}{4E_{\mathbf{k}}^3} \tag{3.8a}$$

$$a_1 = -\frac{1}{\Omega}\sum_{\mathbf{k}}\frac{\xi_{\mathbf{k}}}{4E_{\mathbf{k}}^3} \tag{3.8b}$$

$$a_2 = \frac{1}{\Omega}\sum_{\mathbf{k}}\frac{1}{32m}\left\{\frac{\xi_{\mathbf{k}}(2\xi_{\mathbf{k}}^2 - \Delta^2)}{E_{\mathbf{k}}^5} + \frac{k^2}{dm}\frac{\Delta^2(8\xi_{\mathbf{k}}^2 + 3\Delta^2)}{E_{\mathbf{k}}^7}\right\} \tag{3.8c}$$

$$a_3 = -\frac{1}{\Omega}\sum_{\mathbf{k}}\frac{2\xi_{\mathbf{k}}^2 + \Delta^2}{16E_{\mathbf{k}}^5} \tag{3.8d}$$

and

$$b_0 = a_0 \tag{3.9a}$$

$$b_2 = \frac{1}{\Omega}\sum_{\mathbf{k}}\frac{1}{32m}\left\{-\frac{3\xi_{\mathbf{k}}\Delta^2}{E_{\mathbf{k}}^5} + \frac{k^2}{dm}\frac{\Delta^2(2\xi_{\mathbf{k}}^2 - 3\Delta^2)}{E_{\mathbf{k}}^7}\right\} \tag{3.9b}$$

$$b_3 = \frac{1}{\Omega}\sum_{\mathbf{k}}\frac{\Delta^2}{16E_{\mathbf{k}}^5} \tag{3.9c}$$

(d being the dimensionality). In particular, to determine ξ_{phase} one has to consider the quantity[10]

$$A(\mathbf{q},\omega=0) + B(\mathbf{q},\omega=0) = 2a_0 + (a_2 + b_2)\mathbf{q}^2 + \ldots \tag{3.10}$$

from which

$$\xi_{phase}^2 = \frac{a_2 + b_2}{2a_0} \tag{3.11}$$

To determine the sound velocity s one has instead to consider the full determinant

$$A(\mathbf{q},\omega)A(\mathbf{q},-\omega) - B(\mathbf{q},\omega)^2 =$$
$$2a_0(a_2 - b_2)\mathbf{q}^2 + [2a_0(a_3 - b_3) - a_1^2]\omega^2 + \ldots \tag{3.12}$$

which vanishes for $\omega = \omega(\mathbf{q}) = s|\mathbf{q}|$, with

$$s^2 = \frac{2a_0(a_2 - b_2)}{2a_0(b_3 - a_3) + a_1^2} \tag{3.13}$$

We are left with evaluating the integrals entering Eqs. (3.11) and (3.13). In *three* dimensions we obtain:

$$a_0 = \frac{m}{(2\pi)^2}(2m\Delta)^{1/2}I_5(x_o), \tag{3.14}$$

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$$a_1 = -\frac{m}{(2\pi)^2}\left(\frac{2m}{\Delta}\right)^{1/2} I_6(x_o) \quad , \tag{3.15}$$

$$a_2 + b_2 = \frac{1}{(2\pi)^2}\frac{1}{4}\left(\frac{2m}{\Delta}\right)^{1/2}\left[\int_0^\infty dx\frac{x^2\xi_x^3}{E_x^5} - 2\int_0^\infty dx\frac{x^2\xi_x}{E_x^5} + \frac{10}{3}\int_0^\infty dx\frac{x^4\xi_x^2}{E_x^7}\right] \quad , \tag{3.16}$$

$$a_2 - b_2 = \frac{1}{(2\pi)^2}\frac{1}{4}\left(\frac{2m}{\Delta}\right)^{1/2}\left[I_6(x_o) + 2\int_0^\infty dx\frac{x^4}{E_x^5}\right] \quad , \tag{3.17}$$

$$b_3 - a_3 = \frac{1}{(2\pi)^2}\frac{1}{4}\left(\frac{2m}{\Delta}\right)^{3/2} I_5(x_o) \quad , \tag{3.18}$$

where use has been made of the notation (2.7) and of the integrals (2.20) and (2.21). The four new integrals appearing in Eqs. (3.16) and (3.17) can also be expressed as linear combinations of $I_5(x_o)$ and $I_6(x_o)$. Integrations by parts and simple manipulations lead to:

$$\int_0^\infty dx\frac{x^2\xi_x}{E_x^5} = \frac{1}{6}\int_0^\infty\frac{1}{E_x^3} = \frac{1}{6(1+x_o^2)}\left[\int_0^\infty dx\frac{x^4}{E_x^3}\right.$$
$$\left. - 2\int_0^\infty dx\frac{x^2\xi_x}{E_x^3} + \int_0^\infty dx\frac{1}{E_x}\right]$$
$$= \frac{x_o I_5(x_o) + I_6(x_o)}{6(1+x_o^2)} \quad ; \tag{3.19}$$

$$\int_0^\infty dx\frac{x^2\xi_x^3}{E_x^5} = I_6(x_o) - \int_0^\infty dx\frac{x^2\xi_x}{E_x^5} \quad ; \tag{3.20}$$

$$\int_0^\infty dx\frac{x^4}{E_x^5} = \int_0^\infty dx\frac{x^2\xi_x}{E_x^5} + x_0\int_0^\infty dx\frac{x^2}{E_x^5}$$
$$= (1+x_o^2)\int_0^\infty dx\frac{x^2\xi_x}{E_x^5} + \frac{x_0}{2}I_5(x_o) \quad ; \tag{3.21}$$

$$\int_0^\infty dx\frac{x^4\xi_x^2}{E_x^7} = \frac{3}{10}\int_0^\infty dx\frac{x^2\xi_x}{E_x^5} + \frac{1}{5}\int_0^\infty dx\frac{x^4}{E_x^5} \quad . \tag{3.22}$$

In conclusion we obtain:

$$a_2 + b_2 = \frac{1}{(2\pi)^2}\frac{1}{12}\left(\frac{2m}{\Delta}\right)^{1/2}$$
$$\times \left\{2I_6(x_o) + \frac{(1+4x_o^2)}{3(1+x_o^2)}[I_6(x_o) + x_o I_5(x_o)]\right\} \tag{3.23}$$

and

$$a_2 - b_2 = \frac{1}{(2\pi)^2}\frac{1}{3}\left(\frac{2m}{\Delta}\right)^{1/2}\{I_6(x_o) + x_o I_5(x_o)\} \quad . \tag{3.24}$$

Equation (3.11) thus reduces to

$$(k_F\xi_{phase})^2 = \frac{\epsilon_F}{\Delta}\frac{1}{12I_5(x_o)}$$
$$\times \left\{2I_6(x_o) + \frac{(1+4x_o^2)}{3(1+x_o^2)}[I_6(x_o) + x_o I_5(x_o)]\right\} \tag{3.25}$$

with ϵ_F/Δ given by Eq. (2.27), while Eq. (3.13) becomes

$$\left(\frac{s}{v_F}\right)^2 = \frac{1}{3}\frac{\Delta}{\epsilon_F}\frac{I_5(x_o)(I_6(x_o) + x_oI_5(x_o))}{I_5(x_o)^2 + I_6(x_o)^2} \tag{3.26}$$

where $v_F = k_F/m$.

The expressions (3.25) and (3.26) provide the desired *analytic* expressions of $k_F\xi_{phase}$ and s/v_F for *all* values of x_o. Using again Eqs. (2.24) and (2.26) and the tabulated values of the elliptic integrals, we report in Figs. 4 and 5 the values of $k_F\xi_{phase}$ and s/v_F, respectively, *vs* $k_F\xi_{pair}$.
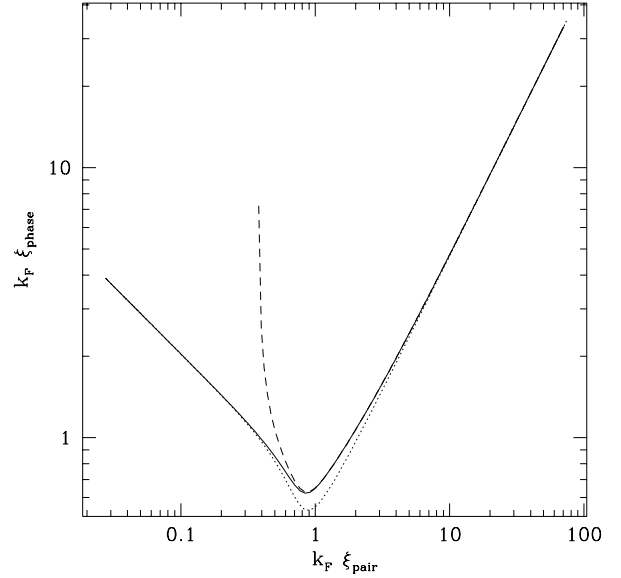

FIG. 4. $k_F\xi_{phase}$ *vs* $k_F\xi_{pair}$, obtained from Eqs. (2.30) and (3.25). Conventions are as in Fig. 1.

To within numerical accuracy, we reproduce in this way the numerical results of Ref. 10 for $k_F\xi_{phase}$. Note, again, that the BE approximation (dotted curve) is surprisingly accurate even on the BCS side.

In the BCS and BE limits one can use the approximate values (2.33) and (2.35), in the order, for the integrals $I_5(x_o)$ and $I_6(x_o)$. This gives:

$$\begin{cases} k_F\xi_{phase} \simeq x_o/3 \\ s/v_F \simeq 1/\sqrt{3} \end{cases} \tag{3.27}$$

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in the BCS limit, and

$$\begin{cases} k_F\xi_{phase} & \simeq (3\pi|x_o|/16)^{1/3} \\ s/v_F & \simeq (12\pi|x_o|)^{-1/3} \end{cases} \qquad (3.28)$$

in the BE limit. Note that in the BE limit the product

$$s\,\xi_{phase} = \frac{1}{m}\left(\frac{s}{v_F}\right)(k_F\xi_{phase}) \simeq \frac{1}{4m} \qquad (3.29)$$

is a constant and coincides with the Bogoliubov result $(2m_B)^{-1}$ for composite bosons with mass $m_B = 2m$, thus confirming the general results established in Ref. 10 for the mapping onto a bosonic system in the strong-coupling limit.
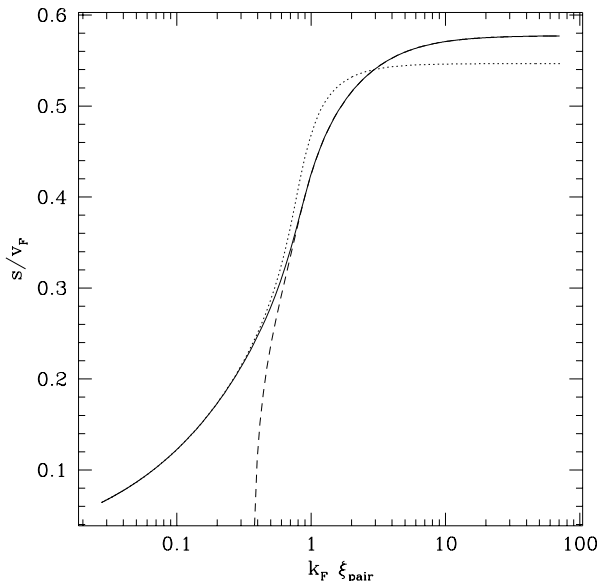

FIG. 5. s/v_F vs $k_F\xi_{pair}$, obtained from Eqs. (2.30) and (3.26). Conventions are as in Fig. 1.

An additional quantity which can be evaluated analytically at the Gaussian level is the coefficient γ of the quartic term in the dispersion relation

$$\omega(\mathbf{q})^2 = s^2\mathbf{q}^2 + \gamma\left(\frac{\mathbf{q}^2}{4m}\right)^2 \quad , \qquad (3.30)$$

obtained by expanding the determinant (3.12) to higher order. Its expression in rather involved and will not be reported here. It is nonetheless interesting to note that in the BCS limit

$$\gamma \simeq -\frac{256}{135}(k_F\xi_{pair})^2 \qquad (3.31)$$

is negative (and large), so that the dispersion relation

$$\omega(\mathbf{q})^2 \simeq \frac{v_F^2\mathbf{q}^2}{3}\left[1 - \frac{1}{3}\mathbf{q}^2\xi_{pair}^2\right] \qquad (3.32)$$

holds for $|\mathbf{q}|$ smaller then a critical value $q_c \propto 1/\xi_{pair}$. This implies that in the BCS limit the wavelength of the collective mode associated with the symmetry breaking cannot be smaller than the size of a Cooper pair.

The above results hold in three dimensions. In two dimensions both ξ_{phase} and s can be expressed in terms of elementary integrals, as shown in Appendix B.

IV. CONCLUDING REMARKS

In this paper we have provided the analytic solution of the crossover problem from BCS to BE in the three-dimensional case for a system of fermions interacting via an attractive contact interaction in free space, at the mean-field level and with the inclusion of Gaussian fluctuations. Although the assumptions required to obtain our analytic solution might be oversimplified in applications to realistic systems, it could be interesting yet to compare our analytic solution with numerical calculations describing more realistic cases. In particular, besides adopting a more sensible momentum-dependent form of the interaction potential, the free one-particle dispersion relation ougth to be replaced by the actual band structure of the medium.

In addition, a detailed description of the crossover problem from BCS to BE would require one to introduce already at the mean-field level the coupling with the charge degrees of freedom, whose effects are expected to be especially important in the crossover region of interest, intermediate between these two limits.[16]

In spite of these limitations, and considering the fact that analytic results for the crossover problem from BCS to BE were thus far limited to the two-dimensional case or to the two limits, our analytic solution is useful as it enables one to describe the crossover region in a compact way with very limited effort.

ACKNOLEDGEMENTS

APPENDIX A: RELEVANT PROPERTIES OF ELLIPTIC INTEGRALS

In this Appendix we briefly review the properties of elliptic integrals that are relevant to our treatment.

The elliptic integrals of the first and second kinds with modulus κ are defined by[12,17]

$$F(\alpha, \kappa) = \int_0^\alpha d\varphi \frac{1}{\sqrt{1 - \kappa^2\sin^2\varphi}} \qquad (A1)$$

$$E(\alpha, \kappa) = \int_0^\alpha d\varphi\sqrt{1 - \kappa^2\sin^2\varphi} \qquad (A2)$$

with $\kappa^2 < 1$. They satisfy the properties

$$F(n\pi, \kappa) = 2nF(\frac{\pi}{2}, \kappa) \qquad (A3)$$

$$E(n\pi, \kappa) = 2nE(\frac{\pi}{2}, \kappa) \qquad (A4)$$

(n integer), and are said to be *complete* when $\alpha = \pi/2$. The complete elliptic integrals admit the following series representations:

$$F(\frac{\pi}{2}, \kappa) = \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 \kappa^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \kappa^4 + \ldots + \left(\frac{(2n-1)!!}{2^n n!}\right)^2 \kappa^{2n} + \ldots \right\} \qquad (A.5)$$

$$E(\frac{\pi}{2}, \kappa) = \frac{\pi}{2} \left\{ 1 - \left(\frac{1}{2}\right)^2 \kappa^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{\kappa^4}{3} - \ldots - \left(\frac{(2n-1)!!}{2^n n!}\right)^2 \frac{\kappa^{2n}}{2n-1} - \ldots \right\} \qquad (A.6)$$

$$F(\frac{\pi}{2}, \kappa) = \ln\frac{4}{\kappa'} + \left(\frac{1}{2}\right)^2 \left(\ln\frac{4}{\kappa'} - \frac{2}{1\cdot 2}\right) \kappa'^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \left(\ln\frac{4}{\kappa'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4}\right) \kappa'^4 + \ldots \qquad (A.7)$$

$$E(\frac{\pi}{2}, \kappa) = 1 + \frac{1}{2}\left(\ln\frac{4}{\kappa'} - \frac{1}{1\cdot 2}\right)\kappa'^2 + \frac{1^2\cdot 3}{2^2\cdot 4}\left(\ln\frac{4}{\kappa'} - \frac{2}{1\cdot 2} - \frac{1}{3\cdot 4}\right)\kappa'^4 + \ldots \qquad (A.8)$$

where $\kappa' = \sqrt{1-\kappa^2}$ is known as the complementary modulus. The representations (A.5) and (A.6) are to be preferred when $\kappa^2 \ll 1$; when $\kappa^2 \simeq 1$ and $\kappa'^2 \ll 1$ the representations (A.7) and (A.8) are to be preferred instead.

Equations (2.24) and (2.26) of the text are obtained by adapting tabulated results and using the properties (A3) and (A4). We obtain:[18]

$$\int_0^\infty dx \frac{1}{\sqrt{x^4 + 2b^2 x^2 + a^4}} = \frac{1}{2a}F(\pi, \kappa) = \frac{1}{a}F(\frac{\pi}{2}, \kappa) \qquad (A.9)$$

$$\int_0^\infty dx \frac{x^2}{\sqrt{x^4 + 2b^2 x^2 + a^4}} = \frac{aE(\pi, \kappa)}{2(a^4 - b^4)} - \frac{F(\pi, \kappa)}{4a(a^2 - b^2)}$$
$$= \frac{a}{a^4 - b^4}E(\frac{\pi}{2}, \kappa) - \frac{1}{2a(a^2 - b^2)}F(\frac{\pi}{2}, \kappa) \qquad (A.10)$$

where $a^2 > b^2 > -\infty$, $a^2 > 0$, and $\kappa^2 = (a^2 - b^2)/(2a^2)$. Comparison of Eqs. (A.9) and (A.10) with Eqs. (2.24) and (2.26) leads us to identify

$$b^2 = -x_o \quad , \qquad a^4 = 1 + x_o^2 \quad , \qquad \kappa^2 = \frac{\sqrt{1+x_o^2} + x_o}{2\sqrt{1+x_o^2}} \qquad (A.11)$$

and results eventually into the right-hand sides of Eqs. (2.24) and (2.26).

APPENDIX B: TWO-DIMENSIONAL CASE

The analytic solution for the two-dimensional case has been already provided in Ref. 11. Its derivation is simpler than for the three-dimensional counterpart obtained in this paper, since it can be expressed in terms of elementary integrals. For the sake of comparison, we report in this Appendix the two-dimensional solution on equal footing of the three-dimensional solution discussed in the text.

Quite generally, in two dimensions the bound-state energy ϵ_o exists for any value of the interaction strength g. For the contact potential we are considering, however, the bound-state equation

$$-\frac{1}{g} = \frac{1}{\Omega}\sum_{\mathbf{k}}\frac{1}{\mathbf{k}^2/2 + \epsilon_o} = \frac{m}{2\pi}\int_0^\infty dy \frac{1}{2y + \epsilon_o/\Delta} \qquad (B.1)$$

[with $y = k^2/(2m\Delta)$] needs to be suitably regularized by introducing an ultraviolet cutoff. This cutoff can, in turn, be removed from further consideration by combining Eq. (B.1) with the gap equation (2.2), namely,

$$-\frac{1}{g} = \frac{1}{2\Omega}\sum_{\mathbf{k}}\frac{1}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}} = \frac{m}{4\pi}\int_{-x_o}^\infty dz \frac{1}{(1 + z^2)^{1/2}} \qquad (B.2)$$

[with $z = y - x_o$ and x_o given by Eq. (2.7)] which requires an analogous regularization. Performing the elementary integrations in Eqs. (B.1) and (B.2) one obtains

$$\frac{\epsilon_o}{\Delta} = \sqrt{1 + x_o^2} - x_o \quad . \qquad (B.3)$$

The normalization condition (2.3) gives further

$$n = \frac{m\Delta}{2\pi}\int_{-x_o}^\infty dz \left(1 - \frac{z}{\sqrt{1+z^2}}\right) = \frac{m\Delta}{2\pi}\left(x_o + \sqrt{1+x_o^2}\right) \quad . \qquad (B.4)$$

Since in the normal state $n = k_F^2/(2\pi) = m\epsilon_F/\pi$, Eq. (B.4) reads

$$\frac{\Delta}{\epsilon_F} = \frac{2}{x_o + \sqrt{1+x_o^2}} \quad . \qquad (B.5)$$

Multiplying at this point both sides of Eqs. (B.3) and (B.5) yields

$$\frac{\epsilon_o}{\epsilon_F} = 2\frac{\sqrt{1+x_o^2}-x_o}{\sqrt{1+x_o^2}+x_o} \quad . \quad (B.6)$$

Finally, the pair-correlation length can be obtained from its definition [cf. Eq. (2.13) for the three-dimensional case]:

$$\xi_{pair}^2 = \frac{1}{m^2}\frac{\int_0^\infty dk(k^3\xi_k^2/E_k^6)}{\int_0^\infty dk(k/E_k^2)} = \frac{2}{m\Delta}\frac{\int_0^\infty dy(y\xi_y^2/E_y^6)}{\int_0^\infty dy(1/E_y^2)} \quad (B.7)$$

with the dimensionless quantities

$$\begin{cases} y = k^2/(2m\Delta) \quad , \quad x_o = \mu/\Delta \quad , \\ \xi_y = y - x_o \quad , \qquad E_y = \sqrt{\xi_y^2+1} \end{cases} \quad (B.8)$$

[in the place of (2.7) for the three-dimensional case]. The integrals in Eq. (B.7) are again elementary and give:

$$(k_F\xi_{pair})^2 = \frac{1}{2}\frac{\epsilon_F}{\Delta}\left[x_o + \left(\frac{2+x_o^2}{1+x_o^2}\right)\left(\frac{\pi}{2}+\arctan x_o\right)^{-1}\right] \quad . \quad (B.9)$$

It is then clear that Δ/ϵ_F, $\mu/\epsilon_F = x_o\Delta/\epsilon_F$, ϵ_o/ϵ_F, and $k_F\xi_{pair}$ can be expressed as functions of x_o. [Note that no reference to the scattering amplitude a_s has been given in two dimensions].

Alternatively, Eq. (B.9) with ϵ_F/Δ given by Eq. (B.5) can be inverted to express x_o (as well as all other quantities) as a function of $k_F\xi_{pair}$.

At the Gaussian level, all integrals entering the definitions (3.8) and (3.9) of the coefficients of the expansions (3.6) and (3.7) are elementary in *two* dimensions. We obtain:

$$a_0 = \frac{m}{8\pi}\frac{\sqrt{1+x_o^2}+x_o}{\sqrt{1+x_o^2}} \quad (B.10)$$

$$a_1 = -\frac{m}{8\pi}\frac{1}{\Delta\sqrt{1+x_o^2}} \quad (B.11)$$

$$a_2 + b_2 = \frac{1}{48\pi}\frac{1}{\Delta}\left\{x_o + \frac{x_o^4+3x_o^2+1}{(1+x_o^2)^{3/2}}\right\} \quad (B.12)$$

$$a_2 - b_2 = \frac{1}{16\pi}\frac{\sqrt{1+x_o^2}+x_o}{\Delta} \quad (B.13)$$

$$b_3 - a_3 = \frac{m}{16\pi}\frac{1}{\Delta^2}\frac{\sqrt{1+x_o^2}+x_o}{\sqrt{1+x_o^2}} \quad . \quad (B.14)$$

These results give

$$(k_F\xi_{phase})^2 = \frac{1}{6}\frac{\epsilon_F}{\Delta}\frac{\sqrt{1+x_o^2}}{\sqrt{1+x_o^2}+x_o}\left\{x_o + \frac{x_o^4+3x_o^2+1}{(1+x_o^2)^{3/2}}\right\} \quad (B.15)$$

and

$$\left(\frac{s}{v_F}\right)^2 = \frac{1}{4}\frac{\Delta}{\epsilon_F}\left(\sqrt{1+x_o^2}+x_o\right) \quad . \quad (B.16)$$

Note that, owing to Eq. (B.5), $(s/v_F)^2 = 1/2$ is independent from x_o. On the other hand, for $k_F\xi_{phase}$ we obtain in the two limits:

$$k_F\xi_{phase} \simeq \begin{cases} x_o/\sqrt{6} & \text{BCS limit} \\ 1/\sqrt{8} & \text{BE limit} \end{cases} , \quad (B.17)$$

thus confirming that the BE limit depends markedly on dimensionality and shows a peculiar behavior in two dimensions.[10] Note, however, that in the BE limit the product $s\,\xi_{phase}$ still coincides with the Bogoliubov result $(4m)^{-1}$.

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